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THREE-DIMENSIONAL SUBSONIC FLOWS AND THEIR BOUNDARY VALUE PROBL--ETC(U)

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THREE-DIMENSIONAL SUBSONIC FLOWS AND THEIR BOUNDARY  
VALUE PROBLEMS EXTENDED TO HIGHER DIMENSIONS

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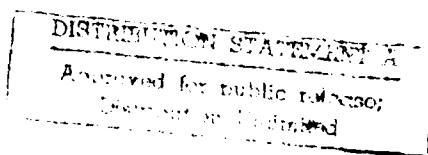
ABSTRACT

In this paper, the steady, irrotational, subsonic flow of a gas around a given profile is studied in the case of arbitrary space dimension greater than two. We prove that the solution of this problem exists, is unique, and depends continuously on the incoming flow. This extends the previous results of Bers and of Finn and Gilbarg.

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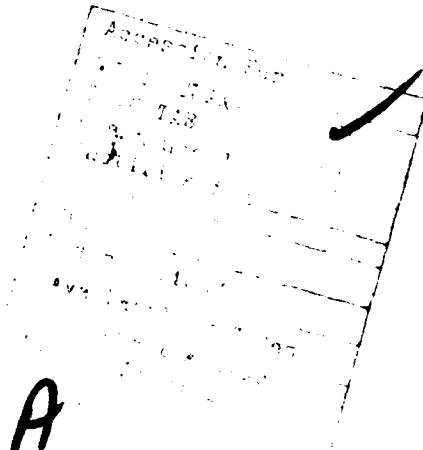


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## SIGNIFICANCE AND EXPLANATION

It is clear that the study of the gas flow around a body moving at subsonic speed is of primary importance in aerodynamics. Extensive studies have been done on the problem of existence and uniqueness of steady irrotational subsonic flow of a perfect gas past a given profile. Bers (1954) treats the plane subsonic flow using the theory of quasianalytic functions. Finn and Gilbarg (1957) deal with three dimensional flows with Mach number less than 0.7. In this paper we prove the existence and uniqueness of three dimensional flow by using and improving a priori estimates obtained earlier by several authors. Our results can also be extended to higher space dimensions. Furthermore we allow the flow to be arbitrarily close to sonic speed.



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THREE-DIMENSIONAL SUBSONIC FLOWS AND THEIR BOUNDARY  
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Guang-Chang Dong\*

The existence and uniqueness of steady irrotational subsonic flow of a perfect gas past a given profile has been studied extensively in the two dimensional case. In [1], Bers proved the existence and uniqueness of plane subsonic flow around a given profile. For higher space dimensions, few results have been previously obtained. Finn and Gilbarg [2] proved existence and uniqueness in three dimensions provided the velocity was not too large (the maximum Mach number less than 0.7).

In this paper we use the idea of [2] together with an improved a priori estimate (extending the method of [4]) to prove the existence and uniqueness of the solution in the three dimensional case. We also extend the result to higher dimensions.

In the following we always suppose the dimension of space is  $n$  ( $n > 3$ ), and use the summation convention

$$a_i x_i = \sum_{i=1}^n a_i x_i ,$$

and denote the vector  $(x_1, \dots, x_n)$  by  $x$ .

The steady irrotational gas flow in  $n$  dimensional space can be described by the velocity potential  $\varphi(x)$ , satisfying the equation

$$\frac{\partial}{\partial x_i} \left( \rho \frac{\partial \varphi}{\partial x_i} \right) = 0 \quad (1)$$

where  $\rho$  represents the density of gas, which is a given positive function of velocity  $q$ , where

$$q = \sqrt{\sum u_i^2}, \quad u_i = \frac{\partial \varphi}{\partial x_i}$$

(1) can be written as

$$a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} = 0 \quad (2)$$

where

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$$a_{ij} = \rho \delta_{ij} + \frac{\rho'}{q} u_i u_j, \quad \delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases} \quad (3)$$

Because (1) and (2) are rotational invariant, the eigenvalues  $\lambda_1, \dots, \lambda_n$  of the quadratic form

$$a_{ij} a_i a_j$$

can be obtained by letting  $u_1 = q, u_2 = \dots = u_n = 0$ , i.e.

$$\lambda_1 = \rho + \rho' q, \quad \lambda_2 = \dots = \lambda_n = \rho \quad .$$

So that if

$$\rho + \rho' q > 0 \quad (4)$$

at this point the quasilinear equation is of elliptic type, and the flow (1) is subsonic at this point. If in some region  $\Omega$  we have

$$\inf_{\Omega} (\rho + \rho' q) > 0 \quad ,$$

then the flow (1) is a subsonic flow in  $\Omega$ .

Assume that the function  $\rho = \rho(q)$  is a positive, three times continuous differentiable functions, defined in the interval

$$0 < q < q_{\text{lim}} < \infty \quad ,$$

and assume that a number  $q_c$  exists (we called it the critical velocity)  $0 < q_c < q_{\text{lim}}$ , such that

$$\rho + \rho' q = \frac{d}{dq} (\rho q) > 0 \quad (0 < q < q_c) \quad . \quad (5)$$

For definiteness, assume that when  $q > q_c$ , the left hand side of (4) is non-positive.

Hence (2) is subsonic if and only if  $0 < q < q_c$ . We also assume that

$$\rho'(0) = 0 \quad . \quad (6)$$

In gas dynamics, the density of a perfect gas is given by the dimensionless formula

$$\rho = \left(1 - \frac{\gamma-1}{2} q^2\right)^{\frac{1}{\gamma-1}} \quad (1 < \gamma < 2, \gamma \text{ is a constant})$$

so

$$\rho + \rho' q = \left(1 - \frac{\gamma-1}{2} q^2\right)^{\frac{2-\gamma}{\gamma-1}} \left(1 - \frac{\gamma+1}{2} q^2\right)$$

hence

$$q_c = \sqrt{\frac{2}{\gamma+1}}, q_{\lim} = \sqrt{\frac{2}{\gamma-1}} .$$

The flow is called subsonic when  $0 < q < q_c$ , which agrees with the discussion above.

The profile  $\Gamma$  is a bounded closed surface in  $n$  dimensional space (it can be some closed surface also). Assume that a constant  $\tau_0$  exists such that  $0 < \tau_0 < 1$  and

$$\Gamma \in C^{(2+\tau_0)} . \quad (7)$$

The region outside  $\Gamma$  (not containing the point at infinity) is denoted by  $\Omega$ . We only consider those satisfying the condition: any closed curve in  $\Omega$  can be deformed into a point without touching  $\Gamma$ .

The flow around  $\Gamma$  means that the solution of (1) in  $\Omega$  satisfies two conditions as follows. The boundary condition

$$\frac{\partial \varphi}{\partial N} \Big|_{\Gamma} = 0 \quad (N \text{ is the interior normal of } \Omega \text{ on } \Gamma) \quad (8)$$

and the condition of uniform flow at  $\Omega$  (called uniform incoming flow)

$$\text{grad } \varphi \Big|_{x=\infty} = u^{\infty} \quad (u^{\infty} \text{ is a constant vector}) . \quad (9)$$

Usually for the problem of flow around a given profile we give the magnitude and direction of incoming flow. Without loss of generality we can assume that the direction of incoming flow is the positive  $x_1$  axis (otherwise after an axis rotation we can reach this situation), i.e.

$$u^{\infty} = (U, 0, \dots, 0) \quad \text{the const } U > 0 \text{ is given} . \quad (10)$$

A slightly different problem is: Given the direction of incoming flow but not the magnitude, i.e.

$$u^{\infty} = (U, 0, \dots, 0) \quad U > 0 \text{ is not given} \quad (11)$$

and given the maximum value of speed instead, i.e.

$$\sup_{\Omega} q = Q \text{ is given} . \quad (12)$$

We shall study the existence and uniqueness, find the flow and other properties of the solution of the above two kinds of flow problem in the subsonic range, i.e. when

$$Q < q_c . \quad (13)$$

#### §1. Preliminary study of the linear problem

Consider the following problem

$$L(\psi) = b_{ij}(x) \frac{\partial^2 \psi}{\partial x_i \partial x_j} + b_i(x) \frac{\partial \psi}{\partial x_i} = f(x) \quad (x \in \Omega, b_{ij} = b_{ji}) \quad (1.1)$$

$$\frac{\partial \psi}{\partial N} \Big|_{\Gamma} = \psi_0 \quad (N \text{ is the interior normal of } \Omega \text{ on } \Gamma) \quad (1.2)$$

$$\psi \Big|_{\infty} = 0 \quad (1.3)$$

where the given functions  $b_{ij}$ ,  $b_i$ ,  $f \in C^{(0)}(\bar{\Omega})$ ,  $\psi_0 \in C^{(0)}(\Gamma)$ , and a positive constant  $\sigma$  exists such that

$$\sigma \sum a_i^2 \leq b_{ij} a_i a_j \leq \frac{1}{\sigma} \sum a_i^2. \quad (1.4)$$

Further conditions on  $b_{ij}$ ,  $b_i$ ,  $f$  are given below. Assuming that the solution of problems (1.1), (1.2), (1.3) exists, and we deduce some a-priori estimates.

Assume that a fixed small positive number  $\tau$  such that  $\tau < \tau_0$  (the definition of  $\tau_0$  see (7)) exists such that

$$|b_{ij}(x) - b_{ij}(\infty)| \leq K_1 r_x^{-\tau} (x \in \Omega) \quad (1.5)$$

$$|b_{ij}(x) - b_{ij}(y)| \leq K_2 r_{xy}^{-\tau} |x-y|^{\tau} (x, y \in \Omega) \quad (1.6)$$

$$|b_i| \leq K_3 r_x^{-1-\tau} \quad (1.7)$$

where the constants  $K_1, K_2, \dots$  depend on  $\sigma, \tau, \Gamma$  only. Moreover, assume  $f$  is a bounded function satisfying

$$|f| \leq K_4 \max_{\bar{\Omega}} |f| r_x^{-n-\tau} \quad (1.8)$$

where

$$r_x = d(x, \Gamma), r_{xy} = \min(r_x, r_y) \quad (1.9)$$

$d(x, \Gamma)$  is the distance between  $x$  and  $\Gamma$ . We have

Lemma 1.

$$|\psi| \leq K_5 \left( \max_{\Gamma} |\psi_0| + \max_{\bar{\Omega}} |f| \right). \quad (1.10)$$

Proof. Let

$$\psi^0 = \max_{\Gamma} |\psi_0| + \max_{\bar{\Omega}} |f|. \quad (1.11)$$

The uniqueness of the solution of (1.1), (1.2), (1.3) is easily deduced from the maximum principle [5]. Thus for  $\psi^0 = 0$  we have  $\psi = 0$ , and in this case (1.10) is true.

When  $\psi^0 \neq 0$ , let

$$L_\infty(\psi) = b_{ij}(\infty) \frac{\partial^2 \psi}{\partial x_i \partial x_j} \quad (1.12)$$

$$R = \sqrt{b_{ij}(\infty) x_i x_j} \quad (1.13)$$

$b_{ij}(\infty)$  is the algebraic complement in the matrix  $(b_{ij}(\infty))$ . We have

$$L_\infty(R^{2-n-\frac{1}{2}}) = \frac{1}{2} (n-2+\frac{1}{2}) R^{-n-\frac{1}{2}}. \quad (1.14)$$

When  $R > K_6$ , where  $K_6$  is a sufficiently large constant, from (1.5), (1.7), (1.8), (1.14) we have

$$L(R^{2-n-\frac{1}{2}}) - \frac{f_0}{\psi^0} = \frac{1}{2} (n-2+\frac{1}{2}) R^{-n-\frac{1}{2}} + O(R^{-n-\tau}) > 0. \quad (1.15)$$

Without loss of generality we can assume that the origin lies inside  $\Gamma$ , then

$\min_{x \in \Omega} R > 0$ . Take  $\alpha = \alpha(\sigma, \tau, \Gamma)$  comparatively large, by using (1.5), (1.7), (1.13) we have

$$\begin{aligned} L(R^{-\alpha}) &= \alpha(\alpha+2) R^{-\alpha-4} b_{ij}(\infty) x_k b_{jh}(\infty) x_h - \alpha R^{-\alpha-2} b_{ij}(\infty) - \\ &\quad - \alpha R^{-\alpha-2} b_{ik}(\infty) x_k > K_1 R^{-\alpha-2}. \end{aligned} \quad (1.16)$$

From (1.15), (1.16) we can select a constant  $K_8$  such that

$$L(R^{2-n-\frac{1}{2}} + K_8 R^{-\alpha}) - \frac{f_0}{\psi^0} > 0 \quad (x \in \Omega). \quad (1.17)$$

Let

$$\left. \begin{array}{l} \psi_1 \\ \psi_2 \end{array} \right\} = \psi + \psi^0 (R^{2-n-\frac{1}{2}} + K_8 R^{-\alpha}). \quad (1.18)$$

From (1.1), (1.17), (1.18) we have

$$L(\psi_1) < 0, \quad L(\psi_2) > 0. \quad (1.19)$$

From (1.3) we have  $\psi_1(\infty) = 0$ , hence  $\min_{\bar{\Omega}} \psi_1 < 0$ , this minimum can only be taken at  $\infty$  or on  $\Gamma$  by (1.19). Similarly  $\max_{\bar{\Omega}} \psi_2 > 0$ , and this maximum can be taken at  $\infty$  or on  $\Gamma$ . Hence we have four cases as follows.

Case 1.  $\min_{\bar{\Omega}} \psi_1 = \psi_1(P_0) < 0$ ,  $P_0 \in \Gamma$ ;  $\max_{\bar{\Omega}} \psi_2 = \psi_2(Q_0) > 0$ ,  $Q_0 \in \Gamma$ .

Extend the method of [6] to prove that (1.10) is true. Because  $\Gamma \in C^{(2)}$ , there exists a positive constant  $\gamma = \gamma(\Gamma)$ , such that for any point  $P \in \Gamma$  we can draw an exterior tangent sphere with radius  $\gamma$  which lies entirely inside  $\Omega$  except point  $P$ .

Draw the exterior tangent sphere with radius  $\gamma$  at  $P_0$ , denote its center by  $P_1$ .

Consider the function

$$\psi_3(P) = \psi_1(P) - \min_{\bar{\Omega}} \psi_1 - k\psi^0(e^{-h\overline{P_1P}^2} - e^{-h\gamma^2}) \quad (P \in \bar{\Omega}) \quad (1.20)$$

where  $k, h$  are positive constants we may choose at our disposal. Take  $h \approx h(\sigma)$  large enough, from (1.17), (1.20) we have

$$L(\psi_3) < 0, \quad \frac{\gamma}{2} < \overline{P_1P} < \gamma. \quad (1.21)$$

From (1.20) we have

$$\psi_3 > 0, \quad \text{when } \overline{P_1P} = \gamma \quad \text{and} \quad \psi_3(P_0) = 0. \quad (1.22)$$

If

$$\psi_3 > 0 \quad \text{when} \quad \overline{P_1P} = \frac{\gamma}{2} \quad (1.23)$$

is true, then from (1.21), (1.22), (1.23) we see that  $\min \psi_3 < 0$  when  $\frac{\gamma}{2} < \overline{P_1P} < \gamma$  can not be true, hence  $\min \psi_3 = \psi_3(P_0) = 0$ . From (1.2), (1.18), (1.20) we have

$$\frac{\partial \psi_3}{\partial N} \Big|_{P_0} = \psi_0(P_0) - \psi^0 \frac{\partial}{\partial N} (R^{2-n-\frac{\tau}{2}} + K_R R^{-\alpha}) \Big|_{P_0} - 2\gamma k h \psi^0 e^{-h\gamma^2} > 0. \quad (1.24)$$

Pick the constant  $K_0$  so that

$$2\gamma K_0 h e^{-h\gamma^2} > 1 + \max \left| \frac{\partial}{\partial N} (R^{2-n-\frac{\tau}{2}} + K_R R^{-\alpha}) \right|$$

then (1.24) can not be true when we take  $K = K_0$ , hence (1.23) can not be true, in other

words, we have  $P_2$  satisfies  $\frac{P_1 P_2}{2} = \frac{Y}{2}$  such that  $\Psi_3(P_2) < 0$ . From (1.20) we have

$$\Psi_1(P_2) - \min_{\bar{\Omega}} \Psi_1 \leq K_{10} \Psi^0, \frac{Y}{2} \leq d(P_2, P_0) \leq \frac{3Y}{2} . \quad (1.25)$$

Similarly there exists a  $Q_2$  satisfying

$$\max_{\bar{\Omega}} \Psi_2 - \Psi_2(Q_2) \leq K_{11} \Psi^0, \frac{Y}{2} \leq d(Q_2, Q_0) \leq \frac{3Y}{2} . \quad (1.26)$$

Because of (1.6) we can apply Harnack's inequality [7] to the nonnegative function

$\Psi_1 - \min_{\bar{\Omega}} \Psi_1$ , and obtain

$$\Psi_1(Q_2) - \min_{\bar{\Omega}} \Psi_1 \leq K_{12} [\Psi_1(P_2) - \min_{\bar{\Omega}} \Psi_1 + \max_{\bar{\Omega}} |L(\Psi_1)|] . \quad (1.27)$$

Combine (1.11), (1.18), (1.25), (1.27) we have

$$\Psi_1(Q_2) - \min_{\bar{\Omega}} \Psi_1 \leq K_{13} \Psi^0 . \quad (1.28)$$

From (1.18), (1.26), (1.28) we have

$$\max_{\bar{\Omega}} \Psi - \min_{\bar{\Omega}} \Psi \leq \max_{\bar{\Omega}} \Psi_2 - \min_{\bar{\Omega}} \Psi_1 \leq (K_{11} + K_{13}) \Psi^0 .$$

From (1.3) we have  $\max_{\bar{\Omega}} \Psi > 0$ ,  $\min_{\bar{\Omega}} \Psi \leq 0$ , so combining the above expression and (1.11) we obtain (1.10).

Case 2.  $\min_{\bar{\Omega}} \Psi_1 = \Psi_1(\infty) = 0$ ,  $\max_{\bar{\Omega}} \Psi_2 = \Psi_2(Q_0) > 0$ ,  $Q_0 \in \Gamma$ .

By lemma 2, there exists positive constants  $R_0(\sigma, \tau, \Gamma)$  and  $K_{14}, K_{15}$  such that the Harnack inequality for the positive function  $R^{2-n} \Psi_1$ , (the definition of  $R$  see (1.13)) for any  $R_1 > R_0$  holds:

$$\max_{R=R_0} (R^{2-n} \Psi_1) \leq K_{14} \min_{R=R_1} (R^{2-n} \Psi_1) + K_{15} \max_{\bar{\Omega}} |f| . \quad (1.29)$$

Proof of (1.29) see lemma 2. From (1.29) we have

$$\begin{aligned} \max_{R=R_0} \Psi_1 &= R_0^{n-2} \max_{R=R_0} (R^{2-n} \Psi_1) \leq K_{14} R_0^{n-2} \max_{R=R_1} (R^{2-n} \Psi_1) + K_{15} R_0^{n-2} \Psi^0 = \\ &= K_{14} \left( \frac{R_0}{R_1} \right)^{n-2} \max_{R=R_1} \Psi_1 + K_{15} R_0^{n-2} \Psi^0 . \end{aligned} \quad (1.30)$$

From (1.19) and  $\Psi_2|_{\infty} = \Psi_1|_{\infty} = 0$ , combining with (1.18) we have

$$\max_{R=R_1} \Psi_1 \leq \max_{R=R_1} \Psi_2 \leq \max_{R=R_0} \Psi_2 \leq \max_{R=R_0} \Psi_1 + K_{16} \Psi^0 \quad (1.31)$$

so combining (1.30), (1.31) we have

$$\left[1 - K_{14} \left(\frac{R_0}{R_1}\right)^{n-2}\right] \max_{R=R_0} \Psi_1 < \left[K_{14} \left(\frac{R_0}{R_1}\right)^{n-2} K_{16} + R_0^{n-2} K_5\right] \Psi^0 .$$

Take  $R_1$  such that  $K_{14} \left(\frac{R_0}{R_1}\right) = \frac{1}{2}$  we have

$$\max_{R=R_0} \Psi_1 < K_{17} \Psi^0 . \quad (1.32)$$

(1.32) is similar to (1.25). We can get (1.26), (1.27), (1.28), (1.10) by the similar process as in case 1.

Case 3.  $\min_{\bar{\Omega}} \Psi_1 = \Psi_1(P_0) < 0 (P_0 \in \Gamma)$ ,  $\max_{\bar{\Omega}} \Psi_2 = \Psi_2(\infty) = 0$ .

Obtaining (1.10) is similar to case 2.

Case 4.  $\min_{\bar{\Omega}} \Psi_1 = 0 = \Psi_1(\infty)$ ,  $\max_{\bar{\Omega}} \Psi_2 = 0 = \Psi_2(\infty)$ .

From (1.18) we have  $\Psi^0 = 0$ , contrary to the hypothesis.

Lemma 1 is thus proved.

Lemma 2. Let

$$\Psi_1 > 0 (x \in \Omega) . \quad (1.33)$$

Prove that (1.29) is true.

Proof. Assume that the region  $R \geq R_{18}$  (the definition of  $R$  see (1.13)) lies in  $\Omega$ . Without loss of generality we can assume

$$\max_{\bar{\Omega}} |f| = 1 \quad (1.34)$$

$$b_{ij}(\infty) = \delta_{ij} . \quad (1.35)$$

Otherwise after a linear change of independent variables and multiplying the unknown

function by a suitable constant we get (1.34) and (1.35). Hence  $R = |x|$ . Apply the

inversion transformation

$$x_i = \frac{x_i}{|x|^2} (i = 1, 2, \dots, n) \quad (1.36)$$

and let

$$\Psi_1 = (R^{2-n} - R^{\frac{2-n-1}{2}}) v \quad (1.37)$$

(1.1) becomes

$$L(v) = B_{ij} v_i v_j + B_i v_i + Bv = F(|x| < \frac{1}{\Psi_{18}}) .$$

Notice that the matrix  $B_{ij}$  is not the same as in (1.23). From (1.1) and (1.36) we see

that  $B_{ij}$  still satisfies the uniform ellipticity condition and (1.33) becomes

$$v > 0 . \quad (1.38)$$

From (1.1), (1.3), (1.5), (1.7), (1.15), (1.18), (1.35), (1.36), (1.37) we have

$$B_{ij} = \delta_{ij} + O(|x|^\tau) \quad (1.39)$$

$$B_i = O(|x|^{\tau-1}) \quad (1.40)$$

$$B = -\frac{\tau}{2}(n-2+\frac{\tau}{2})|x|^{\frac{\tau}{2}-2} + O(|x|^{\tau-2}) \quad (1.41)$$

$$F = -\frac{\tau}{2}(n-2+\frac{\tau}{2})|x|^{\frac{\tau}{2}} + O(|x|^{\tau-2}) \quad (1.42)$$

$$v = O(|x|^{2-\eta}) . \quad (1.43)$$

We use the generalized maximum principle in [8]: When the coefficients satisfy (1.39), (1.40), (1.41),  $w \in C^{(2)}$ ,  $L(w) \geq 0$ ,  $w = 0$  ( $|x|^{2-\eta}$ ) in  $0 < |x| < K_{19}$ , then  $w$  can not attain a positive maximum value in  $0 < |x| < K_{19}$  where  $w(0)$  is defined by

$$\lim_{x \rightarrow 0} w(x).$$

From (1.39) - (1.43) we can take  $v + 2|x|^{\frac{\tau}{2}}$  to be the function  $w$  by taking  $K_{19}$  suitably. Hence  $v$  is bounded above and moreover  $v$  is bounded by combining with (1.38). Then from the result of [8] we get that  $v(0)$  exists and

$$\lim_{x \rightarrow 0} v(x) = v(0) . \quad (1.44)$$

Take positive constant  $K_{20}$  such that the following relations are true when

$0 < |x| < K_{20}$ :

$$L(v) = F < 0, \quad L(v + 2|x|^{\frac{\tau}{2}}) > 0 \quad (1.45)$$

$$L(1) = B < 0, \quad L(1 + 2|x|^{\frac{\tau}{2}}) > 0 . \quad (1.46)$$

We need to determine continuous functions  $K_+(x, y)$ ,  $K_-(x, y)$  in the range  $|x| < K_{20}$ ,  $|y| = K_{20}$  ( $x \neq y$ ) satisfying the following three relations:

$$i. \quad L(K_+) \geq 0, \quad L(K_-) \leq 0 \quad (0 < |x| < K_{20}) \quad (1.47)$$

$$ii. \quad \lim_{x \rightarrow y} \int_0^1 K_{\pm}(x, s) g(s) ds = g(y) \quad (1.48)$$

where  $g(x)$  is any continuous function and  $y_0$  is any fixed point satisfying  $|y_0| = K_{20}$ .

iii. There exists constants  $K_{21}, K_{22}$  such that

$$K_+(X, Y) > K_{21}, K_-(X, Y) < K_{22} \quad (|X| < \frac{K_{20}}{3}, |Y| = K_{20}) \quad (1.49)$$

If  $K_+(X, Y), K_-(X, Y)$  exist, then combine (1.44), (1.45), (1.47), (1.48), (1.49),

and apply the method of [5], (c.f. in §4 of [5]) which extends theorem 1 to the inhomogeneous equations. This is the result that we need. Hence, positive constants  $K_{23}, K_{24}, K_{25}$  exist such that

$$K_{23}v(0) - K_{24} < v(X) < K_{23}^{-1}v(0) + K_{25} \quad (|X| < \frac{K_{20}}{3}) \quad (1.50)$$

(1.29) follows from (1.34), (1.37), (1.50).

We can construct  $K_+$  and  $K_-$  by only altering the method of [5] a little as follows.

Let

$$H = (K_{20}^2 - X^2) [B_{ij}^*(Y)(X_i - Y_i)(X_j - Y_j)]^{-\frac{n}{2}}$$

where  $B_{ij}^*$  are the algebraic complements of  $B_{ij}$ . Take  $K_- = f_-(H)$ , then

$$L(K_-) = B_{ij}(X) H_{X_i X_j} (f_-'' + Af_-') + Bf_-$$

where

$$A = [B_{ij}(X) H_{X_i X_j} + B_i H_{X_i}] / [B_{ij}(X) H_{X_i X_j}] \quad .$$

Apply the estimate method of [5] (the method of obtaining (30) in it) and (1.6),

(1.39) and

$$B_i = 0 (|X|^{\frac{1}{2} - 1}) = 0 (|X-Y| - K_{20})^{\frac{1}{2} - 1}$$

deduced from (1.40), we have

$$|A| < K_{26} (|X-Y|^{n-1+\frac{1}{2}} + |X-Y|^n |X-Y| - K_{20})^{\frac{1}{2} - 1} \quad .$$

And then determining  $f$  etc. (these steps are similar to [5]), at last we get  $K_-(X, Y)$  satisfying (1.47), (1.48), (1.49).

Through the transformation  $K_+ = (1 + 2|X|^2)K_+^*$ , (1.47) becomes

$$L^*(K_+^*) > 0$$

where

$$L^*(v) = B_{ij} v_{X_i X_j} + B_i^* v_{X_i} + B^* v \quad .$$

Applying (1.46) we have

$$B^* = L(1 + 2|X|^2) / (1 + 2|X|^2) > 0 \quad .$$

Let  $K_+ = f_+(H)$ . The selection of  $f_+$  is similar to [5]. The proof of the existence of  $K_+^*$  such that  $K_+^*$  satisfying (1.47), (1.48), (1.49) is similar to  $K_-$ . Hence the proof of lemma 2 is finished.

Lemma 3. We have

$$|\psi| \leq K_{27} \min(1, r_x^{2-n}) (\max_{\Gamma} |\psi_0| + \max_{\Omega} |f|) \quad (1.51)$$

under the conditions (1.4), (1.5), (1.6), (1.7), (1.8).

Proof. Because of lemma 1, we only need to prove the validity of (1.51) when  $r_x$  large enough. From (1.13), (1.14) when  $R \geq K_{28}$  ( $K_{28}$  large enough) we have

$$L(R^{2-n} - R^{2-n-\frac{1}{2}}) - \frac{f}{K_5 \psi^0} = -\frac{1}{2}(n-2+\frac{1}{2})R^{-n-\frac{1}{2}} + 0(R^{-n-\frac{1}{2}}) < 0.$$

Hence the function

$$K_5(R^{2-n} - R^{2-n-\frac{1}{2}})/(K_{28}^{2-n} - K_{28}^{2-n-\frac{1}{2}})\psi^0 \pm \psi \quad (1.52)$$

can not take on a negative minimum value in  $K_{28} < R < \infty$ . From (1.3), (1.10) we see that when  $R = \infty$  and  $R = K_{28}$  (1.52) is nonnegative, hence (1.52) is nonnegative in  $K_{28} < R < \infty$ . This proves the lemma.

Denote the  $k$ 'th order derivatives of  $\psi$  by  $D^{(k)}\psi$ . When  $\psi \in C^{(k)}(\Omega)$  or  $C^{(k+\tau)}(\Omega)$ , let

$$M_{m,k}(\psi) = \text{lub}_{x \in \Omega} r_x^{m+k} |D^{(k)}\psi(x)| \quad (1.53)$$

$$M_{m,k+\tau}(\psi) = \text{lub}_{x,y \in \Omega} r_{xy}^{m+k+2} |D^{(k)}\psi(x) - D^{(k)}\psi(y)| / |x-y|^\tau \quad (1.54)$$

where  $m$  is a nonnegative integral. For the meaning of symbols  $r_x$ ,  $r_{xy}$  see (1.9). The symbol of lub in (1.53), (1.54) means first to take upper bound for all  $k$ 's order derivatives. Let

$$|\psi|_{m,k} = \sum_{j=0}^k M_{m,k}(\psi), |\psi|_{m,k+\tau} = |\psi|_{m,k} + M_{m,k+\tau}(\psi) .$$

When  $\Psi(x) \in C^{(k)}(\Omega)$  and  $|\Psi|_{m,k}$  is finite, denote by  $\Psi(x) \in C_{m,k}$ . Similarly we define  $C_{m,k+\tau}$ .

Lemma 4. Assume that (1.4), (1.6), (1.7) and

$$|b_{ij}(x) - b_{ij}(y)| \leq k_{29} |x-y|^\tau r_{xy}^{-1-\tau} \quad (1.55)$$

are valid, and assume that

$$f \in C_{n,\tau} \quad (1.56)$$

Let  $\Psi \in C^{(2+\tau)}(\Omega)$  be the solution of (1.1), (1.2), (1.3) and  $\Psi \in C_{n-2,0}$ , then we have

$\Psi \in C_{n-2,2+\tau}$ , and

$$|\Psi|_{n-2,2+\tau} \leq k_{30} [M_{n-2,0}(\Psi) + |f|_{n,\tau}] \quad (1.57)$$

Proof. From the interior estimate [9] we obtain (1.57).

Consider a bounded region  $\Omega_0$  satisfying  $\bar{\Omega} \supset \bar{\Omega}_0 \supset \Gamma$ . When  $\Psi \in C^{(k)}(\Omega_0)$  or  $\in C^{(k+\tau)}(\Omega_0)$ , let

$$M_k(\Psi) = \text{lub}_{x \in \Omega_0} |D^{(k)}\Psi(x)|, M_{k+\tau}(\Psi) = \text{lub}_{x,y \in \Omega_0} |D^{(k)}\Psi(x) - D^{(k)}\Psi(y)|/|x-y|^\tau,$$

$$|\Psi|_k = \sum_{j=0}^k M_j(\Psi), |\Psi|_{k+\tau} = |\Psi|_k + M_{k+\tau}(\Psi) \quad .$$

When  $\Psi(x) \in C^{(k)}(\Omega_0)$  and  $|\Psi|_k$  is finite, denote by  $\Psi(x) \in C_k^0$ , similarly we define  $C_{k+\tau}^0$ .

Concerning the function  $\Psi_0$  defined on  $\Gamma$ , let

$$\|\Psi_0\|_k = \text{lub}_{h \leq k} \text{lub}_{x \in \Gamma} |D^{(h)}(\Psi_0)|, \|\Psi_0\|_{k+\tau} = \|\Psi_0\|_k + \text{lub}_{x,y \in \Gamma} |D^{(k)}\Psi_0(x) - D^{(k)}\Psi_0(y)|/|x-y|^\tau$$

where in lub the  $D^{(k)}\Psi_0(x)$  and  $D^{(k)}\Psi_0(y)$  restrict to the derivatives of the same parameter.

Lemma 5. Apart from the hypothesis of lemma 4, let  $b_{ij}$ ,  $b_i$ ,  $f$ ,  $\Psi_0$  satisfy the following conditions

$$|b_{ij}(x) - b_{ij}(y)| \leq k_{31} |x-y|^\tau (x, y \in \Omega_0) \quad (1.58)$$

$$|b_i(x) - b_i(y)| \leq k_{32} |x-y|^\tau \quad (1.59)$$

$$f \in C_\tau^0 \quad (1.60)$$

$$\psi_0 \in C^{(1+\tau)}(\Gamma) . \quad (1.61)$$

And let  $\psi \in C^{(2+\tau)}(\Omega)$  be the solution of (1.1), (1.2), (1.3), then we have  $\psi \in C_{2+\tau}^0$ ,

and

$$\|\psi\|_{2+\tau} \leq K_{33} [M_0(\psi) + \|f\|_{\tau} + \|\psi_0\|_{1+\tau}] . \quad (1.62)$$

Proof. See [10].

Let

$$\bar{M}_{m,k}(\psi) = \text{lub}_{x \in \Omega} \max(1, r_x^{m+k}) |D^{(k)} \psi(x)| ,$$

$$\bar{M}_{m,k+\tau} = \text{lub}_{x,y \in \Omega} \max(1, r_x^{m+k+\tau}) |D^{(k)} \psi(x) - D^{(k)} \psi(y)| / |x-y|^{\tau}$$

$$\|\psi\|_{m,k} = \sum_{j=0}^k M_{m,j}(\psi), \quad \|\psi\|_{m,k+\tau} = \|\psi\|_{m,k} + M_{m,k+\tau}(\psi) .$$

Summing lemmas 3, 4, 5 up, we have

Lemma 6. Under the conditions (1.4), (1.5), (1.6), (1.7), (1.8), (1.55), (1.56), (1.58), (1.59), (1.60), (1.61) let  $\psi \in C^{(2)}(\Omega)$  be the solution of (1.1), (1.2), (1.3), we have

$$\|\psi\|_{n-2,2+\tau} \leq K_{34} (\|f\|_{n,\tau} + \|\psi_0\|_{1+\tau}) . \quad (1.63)$$

Theorem 1. If the conditions of lemma 6 are satisfied then the solution

$\psi \in C^{(0)}(\bar{\Omega}) \cap C^{(2)}(\Omega)$  of (1.1), (1.2), (1.3) exists and is unique.  $\psi \in C^{(2+\tau)}(\Omega)$ , and satisfies the estimate (1.63).

Proof. The uniqueness of the solution follows directly from maximum principle. The existence of the solution can be obtained by applying the continuity method, i.e. to solve the equation with parameter

$$[(1-\theta)L_{\infty} + \theta L]\psi = \theta f \quad (0 < \theta < 1) \quad (1.64)$$

with boundary conditions (1.2), (1.3), where the definition of operator  $L_{\infty}$  is defined by (1.12). When  $\theta = 0$ , through a linear transformation, the equation (1.64), (1.2), (1.3) becomes the Laplace equation in an exterior domain with oblique derivative given. The existence of a solution  $\psi$  and  $\psi \in C^{(2+\tau)}(\Omega)$  can be obtained by the method of integral equations, see [5]. Applying lemma 6 to extend the solution by increasing the parameter  $\theta$  until  $\theta = 1$ .

The theorem is thus proved.

§2. The existence of solution around a given profile.

Theorem 2. The solution of problem (1), (8), (9), (11), (12), (13) exists.

Proof. First we alter the function  $\rho$  a little as follows: Take  $\bar{\rho} = \frac{Q+q}{2}$ , where for the definition of  $q_C$ ,  $Q$  see (5), (12). Let

$$s = s(q) = \frac{1}{2}\{1+\text{th}[(\bar{\rho}-Q)(2q-Q-\bar{\rho})]/[2(q-Q)(\bar{\rho}-q)]\}$$

then  $s(q) \in C^{(\infty)}$  when  $Q < q < \bar{\rho}$  and increases monotonously, and

$$s(Q) = s'(\bar{\rho}) = s''(Q) = \dots = s'(\bar{\rho}) = s''(\bar{\rho}) = \dots = 0, s(\bar{\rho}) = 1.$$

Let

$$\bar{\rho} = \begin{cases} \rho & (0 < q < Q) \\ \rho(1-s) + es & (Q < q < \bar{\rho}) \\ e & (\bar{\rho} < q < \infty) \end{cases} \quad (2.1)$$

where the constant  $e = \max_{Q < q < \bar{\rho}} \rho$ , it is easy to see that the smoothness of  $\bar{\rho}$  is the same as  $\rho$ , i.e.  $\bar{\rho} \in C^{(3)}$ , and  $\bar{\rho} > 0$ . Applying (5) we have

$$\bar{\rho} + \bar{\rho}'q = (\rho + \rho'q)(1-s) + es + q(e-\rho)s' > 0 \quad (Q < q < \bar{\rho}) \quad (2.2)$$

Consider the equation

$$\frac{\partial}{\partial x_i} (\bar{\rho} \frac{\partial \varphi}{\partial x_i}) = 0$$

or

$$\bar{a}_{ij}(D\varphi) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} = 0 \quad (2.3)$$

where

$$\bar{a}_{ij}(D\varphi) = \bar{\rho} \delta_{ij} + \frac{\bar{\rho}'}{q} u_i u_j.$$

The eigenvalues of its correspondent quadratic form  $\bar{a}_{ij}(D\varphi) \alpha_i \alpha_j$  are

$$\lambda_1 = \bar{\rho} + \bar{\rho}'q, \lambda_2 = \dots = \lambda_n = \bar{\rho}.$$

From (2.2) we see that all eigenvalues are positive, i.e. when  $0 < q < \infty$  (2.3) is always an elliptic equation, and a positive constant  $\sigma = \sigma(Q)$  exists such that

$$\sigma \sum \alpha_i^2 \leq \bar{a}_{ij}(D\varphi) \alpha_i \alpha_j \leq \frac{1}{\sigma} \sum \alpha_i^2.$$

Next we prove that the solution of (2.3), (8), (9), (11), (12), (13) exists. From (13) and (2.1) we see that it also is the solution of (1), (8), (9), (11), (12), (13).

Hence concerning the proof of theorem 2, without loss of generality we can assume the quasilinear equation (2) is uniformly elliptic when  $0 < q < \infty$ , i.e. there has positive constant  $\sigma = \sigma(\Omega)$  such that

$$\sigma \sum a_i^2 < a_{ij}(D\varphi) a_i a_j < \frac{1}{\sigma} \sum a_i^2 . \quad (2.4)$$

Let the function  $\varphi(x)$  satisfy

$$\varphi \in C^{(1+\tau)}(\bar{\Omega} \cup \Gamma) \quad (2.5)$$

$$D\varphi(\infty) = (U, 0, \dots, 0)$$

$$[\varphi(x) - Ux_1]_{x=\infty} = 0 \quad (2.6)$$

$$|D\varphi(x) - D\varphi(\infty)| \leq C_1 r_x^{-\tau}$$

$$|D\varphi(x) - D\varphi(y)| \leq C_2 \min(1, r_{xy}^{-\tau}) |x-y|^\tau$$

where

$$\tau = \tau(\Omega, \Gamma) < \tau_0 \quad (\text{the definition of } \tau_0 \text{ see (7)}) \quad (2.7)$$

is a positive constant which shall be determined later,  $U$  is a positive constant not to be fixed, (for the meaning of  $r_x, r_{xy}$  see (1.9)),  $C_1, C_2$  are positive constants. The norm of  $\varphi(x)$  is defined by

$$\|\varphi\| = |U| + \max_{\bar{\Omega}} |\varphi - Ux_1| + \inf C_1 + \inf C_2 .$$

It is easy to see all functions  $\varphi$  form a Banach space  $E$ .

We wish to find the solution of the following equations:

$$a_{ij}(D\varphi) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} = 0 \quad (2.8)$$

$$\frac{\partial \varphi}{\partial N}|_{\Gamma} = -\cos(N, x_1)|_{\Gamma} \quad (2.9)$$

$$\varphi|_{\infty} = 0 . \quad (2.10)$$

Let

$$\Phi(x) = \theta Q[\varphi(x) + x_1] / \max_{\bar{\Omega}} |\text{grad}[\varphi(x) + x_1]| \quad (0 < \theta < 1) \quad (2.11)$$

where the definition of  $Q$  see (12).

By theorem 1 we have  $\Phi \in E$ , hence a functional in  $E$  is defined as follows

$$\psi = T(\varphi, \theta) \quad (0 < \theta < 1) . \quad (2.12)$$

Now use Leray-Schauder degree theory [11] to prove that for any  $0 < \theta < 1$  the solution of functional equation

$$\varphi - T(\varphi, \theta) = 0 \quad (2.13)$$

exists. By theorem 1 and (2.8), (2.9), (2.10), (2.11) we see that when  $\theta = 1$  the solution of (2.12) is a solution of (1), (8), (9), (11), (12), (13).

Let us verify the three conditions of Leray-Schauder degree theory.

i. From theorem 1 we have

$$\|\psi\|_{n-2,2+\tau} \leq K_{34} \|\varphi\| . \quad (2.14)$$

Hence  $T$  is completely continuous.  $T$  is also closed, i.e. when  $\varphi_k \in E$ ,  $\varphi \in E$  and  $\|\varphi_k - \varphi\| \rightarrow 0$  ( $k \rightarrow \infty$ ) the corresponding solutions satisfying  $\|\phi_k - \phi\| \rightarrow 0$ . If this is not true, i.e. there is a subsequence  $\{k'\}$  and positive constant  $\delta$  such that

$$\|\phi_{k'} - \phi\| > \delta . \quad (2.15)$$

When  $k'$  is large, from (2.14) we have

$$\|\psi_{k'}\|_{n-2,2+\tau} \leq K_{34} (1 + \|\varphi\|) ,$$

hence  $\{\psi_{k'}\}$  is compact in  $\|\cdot\|_{n-2,2+\frac{\tau}{2}}$  i.e. there exist  $\psi_*$  and a subsequence  $\{k''\}$  of  $\{k'\}$  such that

$$\|\psi_{k''} - \psi_*\|_{n-2,2+\frac{\tau}{2}} \rightarrow 0 .$$

From  $a_{ij}(D\psi_{k''}) \frac{\partial^2 \psi_{k''}}{\partial x_i \partial x_j} = 0$  taking limits we have

$$a_{ij}(D\psi_*) \frac{\partial^2 \psi_*}{\partial x_i \partial x_j} = 0 .$$

Similarly we have

$$\frac{\partial \psi_*}{\partial N}|_T = -\cos(N, x_1)|_T, \psi_*|_\infty = 0 .$$

Hence from the uniqueness of the solution of (2.8), (2.9), (2.10) we have  $\psi_* = \psi$ , hence

$$\|\phi_{k''} - \phi\| \rightarrow 0 .$$

This contradicts to (2.15), therefore  $T$  is closed.

From (2.11) we see that  $T$  is continuous uniformly in  $\theta$ , hence the degree of (2.13) is independent of  $\theta$ .

ii. When  $A = 0$ ,  $\phi = 0$ , i.e. (2.13) has only the solution  $\varphi = 0$ . Therefore, the degree of (2.13) is equal to 1.

iii. To prove the solution of (2.13) is bounded in  $E$ , in other words, to prove that the solution of (1), (8), (9), (11) and

$$\max_{\bar{\Omega}} q = \theta Q \quad (0 < \theta < 1) \quad (2.16)$$

is bounded in  $E$ , which shall be proved in the following two sections, first notice that from (2.16) we have

$$\max_{\bar{\Omega}} q \leq Q . \quad (2.17)$$

### §3. The interior and boundary estimates of solution.

Let  $\varphi \in E$  satisfy (1), (8), (9), (11), (2.17). Consider (2) as a linear equation in  $\varphi$ , from (2.5) and the Schauder estimate (notice that we have (7) and (2.7)) we have

$$\varphi \in C^{(3+\tau)}(\Omega) \cap C^{(2+\tau)}(\Omega+\Gamma) .$$

Differentiate (1) about  $x_h$  we have

$$\frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) = 0 \quad (u \text{ is the abbreviation of } \frac{\partial \varphi}{\partial x_h} = u_h) \quad (3.1)$$

where the definition of  $a_{ij}$  see (3).

Assume the sphere  $|x-x_0| \leq \mu$  lies in  $\Omega$  and  $0 < \theta < 1$ . Let

$$\zeta(x) = \zeta(x, \mu, \theta) = \begin{cases} 1 & |x-x_0| \leq \mu(1-\theta) \\ \frac{1}{\theta\mu} (u - |x-x_0|) & \mu(1-\theta) \leq |x-x_0| \leq \mu \\ 0 & |x-x_0| > \mu \end{cases} \quad (3.2)$$

Multiply (3.1) by  $\zeta^2(x)u$  and integrate, we have

$$\int \zeta^2 a_{ij} u_i u_j dx = -2 \int a_{ij} u u_i \zeta \zeta_j dx .$$

From this formula and (2.4) we have

$$\sigma \int \zeta^2 (\text{grad } u)^2 dx \leq L_1 \int [\varepsilon \zeta^2 (\text{grad } u)^2 + \frac{1}{\varepsilon} u^2 (\text{grad } \zeta)^2] dx .$$

In this and the following sections the constants  $L_1, L_2, \dots$  depend on  $Q, \Gamma$  only.  $\varepsilon$  is an arbitrary positive constant. Take  $\varepsilon L_1 = \frac{\sigma}{2}$  we have

$$\int \zeta^2 (\text{grad } u)^2 dx \leq L_2 \int u^2 (\text{grad } \zeta)^2 dx . \quad (3.3)$$

Take  $\theta = \frac{1}{2}$  from (3.2), (3.3), (2.17) we have

$$\int_{|x-x_0| \leq \frac{\mu}{2}} (\text{grad } u)^2 dx \leq L_3 u^{n-2} . \quad (3.4)$$

Let  $\eta$  be a constant. Multiply (3.1) by the function

$$\begin{cases} \zeta^2(x)(u-\eta) & (u > \eta) \\ 0 & (u < \eta) \end{cases}$$

and integrate, we obtain similarly

$$\int_{u>\eta} \zeta^2(\operatorname{grad} u)^2 dx \leq L_4 \int_{u>\eta} (u-\eta)^2 (\operatorname{grad} \zeta)^2 dx. \quad (3.5)$$

We denote the sets  $u(x) \geq \eta$  in sphere  $|x-x_0| \leq \mu$  by  $A_{\eta,\mu}$ ,  $B_{\eta,\mu}$  respectively.

From (3.2), (3.5) we have

$$\int_{A_{\eta,\mu-\theta\mu}} (\operatorname{grad} u)^2 dx \leq L_5 \operatorname{mes} A_{\eta,\mu} \frac{1}{(\theta\mu)^2} \max_{x \in A_{\eta,\mu}} [u(x) - \eta]^2. \quad (3.6)$$

Similarly we have

$$\int_{B_{\eta,\mu-\theta\mu}} (\operatorname{grad} u)^2 dx \leq L_6 \operatorname{mes} B_{\eta,\mu} \frac{1}{(\theta\mu)^2} \max_{x \in B_{\eta,\mu}} [\eta - u(x)]^2. \quad (3.7)$$

From (3.4), (3.6), (3.7) and applying the result of [3]: There exist constants  $L$ , and

$\tau_1 = \tau_1(Q, \Gamma)$  ( $0 < \tau_1 < 1$ ) such that for any  $x, y \in \Omega$  we have

$$|u(x) - u(y)| \leq L_7 r_{xy}^{\tau_1} |x-y|^{\tau_1}. \quad (3.8)$$

Note. In [3] the inequality (3.8) is proved under the restriction  $|x-y| \leq L_8 r_{xy}$ . When  $|x-y| > L_8 r_{xy}$ , the validity of (3.8) is true by (2.17).

We now turn to the boundary estimate. From (7) we see that there is a local parametric expression for a little part of  $\Gamma$

$$x_i = x_i(\xi_1, \dots, \xi_{n-1}) \in C^{(2+\tau_0)}$$

in the neighborhood of any point  $P_0 \in \Gamma$ .

Choose variables  $(\xi_1, \dots, \xi_n)$  suitably such that inside the small sphere  $\omega$  with center  $P_0$  we have  $\xi \in C^{(2+\tau_0)}$ , on point  $P_0(\xi_1, \dots, \xi_n)$  for a unit orthogonal axis system, and  $\xi_n$  coincides with the direction of normal on  $\omega \cap \Gamma$ . The method for choosing is as follows: First take  $\xi_1, \dots, \xi_{n-1}$  such that they form the unit orthogonal axis system on  $P_0$ . Denote

$$\cos(N, x_i) = \lambda_i(\xi_1, \dots, \xi_{n-1}) \quad (i = 1, \dots, n)$$

on  $\Gamma$ . Then the inverse function  $\xi_1, \dots, \xi_n$  of

$$x_i = x_i(\xi_1, \dots, \xi_{n-1}) + \xi_n \int_0^1 \lambda_i(\xi_1 + \sigma_1 \xi_n, \dots, \xi_{n-1} + \sigma_{n-1} \xi_n) d\sigma_1 \dots d\sigma_{n-1}$$

satisfies all our requirements.

Let

$$\alpha_{jk} = \frac{\partial \xi_j}{\partial x_k} \frac{\partial \xi_k}{\partial x_i} \quad (3.9)$$

we have

$$\alpha_{jk}|_{\Gamma_0} = \delta_{jk} \quad (j, k = 1, \dots, n) \quad (3.10)$$

$$\alpha_{jn}|_{\omega \cap \Gamma} = 0 \quad (j = 1, \dots, n-1) \quad . \quad (3.11)$$

From (3.9) we have

$$q^2 = \sum u_i^2 = \alpha_{jk} \bar{u}_j \bar{u}_k \quad (\bar{u}_j = \frac{\partial \varphi}{\partial \xi_j}) \quad .$$

Equation (1) is the Euler equation of the variation problem  $\delta \int F(q) dx = 0$  where

$F(q) = \int \rho q dq$ . Under the transformation we have

$$\delta \int F(q) J d\xi = 0 \quad (J = \left| \frac{\partial(x_1, \dots, x_n)}{\partial(\xi_1, \dots, \xi_n)} \right|)$$

or

$$\frac{\partial}{\partial \xi_j} (J \rho \alpha_{ij} \bar{u}_j) = 0 \quad . \quad (3.12)$$

Differentiate (3.12) with respect to  $\xi_h$  we have

$$\frac{\partial}{\partial \xi_j} (J \beta_{ij} \frac{\partial \bar{u}_h}{\partial \xi_j} + \beta_i) = 0 \quad (3.13)$$

where

$$\beta_{ij} = \rho \alpha_{ij} + \frac{\rho'}{q} \alpha_{ik} \alpha_{jm} \bar{u}_k \bar{u}_m \quad (3.14)$$

$$\beta_i = \frac{\partial}{\partial \xi_h} (J \alpha_{ij} \rho \bar{u}_j + 2J \rho' q \alpha_{ij} \frac{\partial \alpha_{km}}{\partial \xi_h} \bar{u}_j \bar{u}_k \bar{u}_m) \quad . \quad (3.15)$$

From (3.10), (3.14) we have

$$\beta_{ij}|_{P_0} = (\rho \delta_{ij} + \frac{\rho'}{q} \bar{u}_i \bar{u}_j)|_{P_0} \quad . \quad (3.16)$$

Hence when the radius of  $\omega$  is sufficiently small, from (2.4), (3.16) we have

$$\frac{\sigma}{2} \sum \lambda_i^2 < \beta_{ij} \lambda_i \lambda_j < \frac{2}{\sigma} \sum \lambda_i^2 . \quad (3.17)$$

Denote the image of  $w, w \cap \Omega, w \cap \Gamma$  by  $\bar{w}, \bar{\Omega}, \bar{\Gamma}$ . (8) becomes

$$u_n|_{\bar{\Gamma}} = 0 . \quad (3.18)$$

Let  $\zeta(\xi, \mu, 0)$  be the function defined by (3.3) with  $x, x_0$  changing to  $\xi, \xi_0$ , restrict in this case that the sphere with center  $\xi_0$ , radius  $\mu$  lies entirely inside  $\bar{w}$ . Multiply (3.13) by

$$\begin{cases} \zeta^2(\bar{u}_h - n) & (\bar{u}_h > n) \\ 0 & (\bar{u}_h < n) \end{cases}$$

and integrate, but in case  $h = n$  restrict  $n > 0$ . Integrating by parts we have

$$\int_{\bar{w} \cap \{\bar{u}_h > n\}} (J\beta_{ij} \frac{\partial \bar{u}_h}{\partial \xi_j} + \beta_{ii}) \frac{\partial}{\partial \xi_i} [\zeta^2(\bar{u}_h - n)] d\xi = 0 . \quad (3.19)$$

Since there is no surface integral in (3.19), we only need to check the case  $i = n$ . When  $h = n$ , on  $\bar{\Gamma}$  by (3.18) we have  $\bar{u}_n - n < 0$ . When  $h \neq n, j = n$  from (3.18) we have

$$\frac{\partial \bar{u}_h}{\partial \xi_n}|_{\bar{\Gamma}} = \frac{\partial \bar{u}_n}{\partial \xi_n}|_{\bar{\Gamma}} = 0 . \quad \text{When } h \neq n, j \neq n, \text{ from (3.11), (3.14) we have } \beta_{nj}|_{\bar{\Gamma}} = 0 . \quad \text{And}$$

the  $b_i$  defined by (3.15) has the property  $b_n|_{\bar{\Gamma}} = 0$  when  $h \neq n$ , the reason is, from

$$(3.11) \text{ we have } \alpha_{nj}|_{\bar{\Gamma}} = 0 \quad (j \neq n), \text{ and this induces } \frac{\partial \alpha_{nj}}{\partial \xi_n}|_{\bar{\Gamma}} = 0 \quad \text{in the case } j \neq n,$$

$h \neq n$ , combine with (3.18) we have  $b_n|_{\bar{\Gamma}} = 0$ .

From (3.19) by applying Schwarz inequality and (3.17) we have

$$\int_{\bar{\Omega} \cap A_{\mu}, \mu=0} (\text{grad } \bar{u}_h)^2 d\xi \leq L_9 \text{mes}(\bar{\Omega} \cap A_{\mu}, \mu) \left( \frac{1}{(\theta\mu)^2} \max_{\xi \in \bar{\Omega} \cap A_{\mu}, \mu} [\bar{u}_h(\xi) - n]^2 + 1 \right) . \quad (3.20)$$

Define

$$\bar{u}_h(\xi_1, \dots, \xi_n) = \begin{cases} \bar{u}_h(\xi_1, \dots, \xi_{n-1}, -\xi_n) & (h \neq n) \\ 0 & (h = n) \end{cases} \quad (3.21)$$

in the part of  $\bar{w}$  outside  $\bar{\Omega}$ , then apply (3.18) we have  $\bar{u}_h \in C^{(0)}(\bar{w}) \cap W_2^1(\bar{w})$ . From (3.20), (3.21) we have

$$\int_{A_{n,\mu-\theta\mu}} (\operatorname{grad} \bar{u}_h)^2 d\xi \leq 2L_9 \operatorname{mes} A_{n,\mu} \left( \frac{1}{(\theta\mu)^2} \max_{\xi \in A_{n,\mu}} [\bar{u}_h(\xi) - n]^2 + 1 \right)$$

(restrict  $n > 0$  when  $h = n$ ).

By the same process we can obtain two formulas similar to (3.4), (3.7).

From the result of [3] we have,  $\bar{u}_h$  ( $h = 1, \dots, n$ ) satisfy a Hölder condition in  $\bar{\omega}$  i.e. there exist  $L_{10}$  and  $\tau_2(\Omega, \bar{\omega})$  ( $0 < \tau_2 < 1$ ) such that for any  $x, y \in \Omega \cap \bar{\omega}$  we have

$$|u(x) - u(y)| \leq L_{10} |x-y|^{\tau_2}. \quad (3.22)$$

#### §4. Estimate of solution in the neighborhood of $\infty$ .

Assume that the solution  $\varphi$  of (1), (8), (9), (11), (2.17) satisfies

$$\varphi \in C^{(2)}(\Omega) \cap C^{(1)}(\Omega + \Gamma). \quad (4.1)$$

When  $\varphi \in E$ , from (2.5) and applying the Schauder interior estimate twice we have

$$\varphi \in C^{(3+\tau)}(\Omega) \cap C^{(1)}(\Omega + \Gamma).$$

Let  $0 < \theta < 1$ ,  $0 < \theta_1 < 1$ ,  $\lambda$  is a positive constant,  $R, \bar{R}$  are large positive numbers. Let

$$\zeta(x) = \zeta(x, R, \bar{R}) = \begin{cases} |x|^\lambda & R(1+\theta) \leq |x| \leq \bar{R}(1-\theta_1) \\ R^{\lambda(1+\theta)} \frac{|x|-R}{\theta R} & R \leq |x| \leq R(1+\theta) \\ \bar{R}^{\lambda(1-\theta_1)} \frac{\bar{R}-|x|}{\theta_1 \bar{R}} & \bar{R}(1-\theta_1) \leq |x| \leq \bar{R}. \end{cases} \quad (4.2)$$

Using this  $\zeta$  we can obtain (3.3) also. When  $\lambda < 1 - \frac{n}{2}$ , from (2.17) and (4.2) we see that (3.3) is also true when  $\zeta = \zeta(x, R, \infty)$ .

Denote the inversion point of  $x$  about the unit sphere by  $X$ , then (3.3) becomes

$$\int \zeta^2 |x|^{4-2n} (\Delta_X u)^2 dx \leq L_2 \int u^2 |x|^{4-2n} (\Delta_X \zeta)^2 dx. \quad (4.3)$$

Take  $\lambda = 2 - n$ ,  $\theta = 1$ ,  $\bar{R} = \infty$  and let  $\frac{1}{R} = u$ , from (4.2) and (4.3) we have

$$\int_{|x| \leq \frac{u}{2}} (\operatorname{grad}_X u)^2 dx \leq L_{11} u^{n-2}. \quad (4.4)$$

Similarly we have

$$\int_{A_{n,\mu(1+u)}^{-1}} (\operatorname{grad}_X u)^2 dx \leq L_{12} \operatorname{mes} A_{n,\mu} \frac{1}{(\theta\mu)^2} \max_{X \in A_{n,\mu}} [u(X) - n]^2 \quad (4.5)$$

$$\int_{B_{n,\mu(1+\theta)}^{-1}} (\operatorname{grad}_X u)^2 dx \leq L_{13} \operatorname{mes} B_{n,\mu} \frac{1}{(\theta\mu)^2} \max_{x \in B_{n,\mu}} |u(x) - n|^2 \quad (4.6)$$

where  $A_{n,\mu}$ ,  $B_{n,\mu}$  are the sets of  $u(x) > n$  in  $|x| < \mu$  respectively.

It is easy to prove that  $u(x) \in W_2^1$  in a neighborhood of the origin (which includes the origin). From (2.17), (4.4), (4.5), (4.6) and applying the result of [3], we see that constants  $\alpha = \alpha(Q, \Gamma) > 0$  and  $L_{14}$  exist such that in the neighborhood of the origin the following inequality is valid:

$$|u(x) - u|_{x=0} \leq L_{14} |x|^\alpha \quad (4.7)$$

where  $u|_{x=0} = u|_{x=\infty} = u(\infty)$  is defined from (11). Let  $u(x) - u(\infty) = v(x)$ , then (4.7) becomes

$$|v(x)| \leq L_{14} |x|^{-\alpha} . \quad (4.8)$$

Apply a linear transformation  $x = x(y)$  such that the matrix  $(a_{ij})_{x=\infty}$  becomes an unit matrix. Then (3.1) becomes

$$\frac{\partial}{\partial y_i} (c_{ij} \frac{\partial v}{\partial y_j}) = 0 \quad (4.9)$$

where  $c_{ij}(\infty) = \delta_{ij}$ . Combining with (4.8) we have

$$|c_{ij}(y) - \delta_{ij}| \leq L_{15} |y|^{-\alpha} . \quad (4.10)$$

Multiply (4.9) by  $\zeta^2 v$  and integrate to obtain

$$\int \zeta^2 (\operatorname{grad} v)^2 dy \leq L_{16} \int v^2 (\operatorname{grad} \zeta)^2 dy \quad (4.11)$$

by using the same method as (3.3) was obtained, where  $\zeta(y) = \zeta(y, R, \bar{R})$  is determined from (4.2) by changing  $x$  to  $y$ . Take  $\beta$  satisfying

$$0 < \beta < \frac{n}{2} - 1, \quad \beta < \alpha . \quad (4.12)$$

In (4.2) take  $\lambda = 1 - \frac{n}{2} + \beta$ ,  $\bar{R} = \infty$ ,  $\theta = 1$ , substituting in (4.11) and applying (4.8) we have

$$f(r) = \int_{|y|>r} |y|^{2\lambda} (\operatorname{grad} v)^2 dy \leq L_{17} r^{2(\beta-\alpha)} . \quad (4.13)$$

Estimate by extending the method of [4]. Denote the unit spherical surface with center origin by  $\omega$ , let  $r < R$ , and denote  $|y| = \mu$  we have

$$\begin{aligned}
f(r) &= \int_{r \leq |y| \leq R} |y|^{2\lambda} (\text{grad } v)^2 dy \geq \int_{\omega} dw \int_{Y}^R \left( \frac{\partial v}{\partial \mu} \right)^2 \mu^{n-1+2\lambda} d\mu \\
&> \int_{\omega} \left[ \int_r^R \frac{\partial v}{\partial \mu} d\mu / \int_r^R \mu^{-n+1-2\lambda} d\mu \right]^2 dw \geq (n-2+2\lambda) r^{n-2+2\lambda} \int_{\omega} [v(R) - v(r)]^2 dw
\end{aligned} \tag{4.14}$$

where in the above expression Schwarz's inequality was used. Let  $R \rightarrow \infty$  in (4.14).

Applying (4.8), (4.13) we have

$$f(r) \geq (n-2+2\lambda) r^{n-2+2\lambda} \int_{\omega} v^2(r) dw = (n-2+2\lambda) \int_{|y|=r} |y|^{2\lambda-1} v^2 ds . \tag{4.15}$$

Combining (4.12), (4.13), (4.15) we have

$$\int_{|y|=r} |y|^{2\beta-2} v^2 dy \leq L_{18} r^{2(\beta-\alpha)} . \tag{4.16}$$

Let

$$g_R(r) = \int_{r \leq |y| \leq R} |y|^{2\lambda} c_{ij} \frac{\partial v}{\partial y_i} \frac{\partial v}{\partial y_j} dy . \tag{4.17}$$

Integrating by parts and applying (4.9) we have

$$\begin{aligned}
g_R(r) &= \left( \int_{|y|=R} - \int_{|y|=r} \right) |y|^{2\lambda} v c_{ij} \frac{\partial v}{\partial y_i} \cos(N, y_j) ds - \\
&\quad \int_{r \leq |y| \leq R} |y|^{2\lambda-1} v c_{ij} \frac{\partial v}{\partial y_j} \cos(N, y_j) dy
\end{aligned} \tag{4.18}$$

where  $N$  is the normal directed toward  $\infty$ . When  $R = \infty$  from (4.10), (4.13) we see that the left hand side of (4.18) has a limit, from (4.13), (4.16) applying Schwarz inequality we prove easily that the last term in the right hand side of (4.18) has a limit when

$R \rightarrow \infty$ , hence when  $R \rightarrow \infty$

$$\int_{|y|=R} |y|^{2\lambda} v c_{ij} \frac{\partial v}{\partial y_i} \cos(N, y_j) ds \tag{4.19}$$

has a limit. If this limit value is a constant  $p \neq 0$ , then applying (4.10), (4.13),

(4.15) when  $\mu$  large enough we have

$$\begin{aligned}
p^2 &\leq L_{18} \left( \int_{|y|=\mu} |y|^{2\lambda} |v| |\text{grad } v| ds \right)^2 \leq L_{19} \mu \int_{|y|=\mu} |y|^{2\lambda-1} v^2 ds \\
&\leq \int_{|y|=\mu} |y|^{2\lambda} (\text{grad } v)^2 ds \\
&\leq [-L_{19} \mu f(\mu) f'(\mu) f''(\mu)] / (n - 2 + 2\lambda) .
\end{aligned}$$

Integrating we get

$$p^2 (n-2-2\lambda) \ln \mu / L_{19} \int_{R_0}^R \leq -\frac{1}{2} f^2(\mu) \int_{R_0}^R \quad (R_0 < R) \tag{4.20}$$

(4.13) contradicts (4.20) when  $R \rightarrow \infty$ , hence the limit of (4.19) must be zero. Let

$R \rightarrow \infty$  in (4.18) and denoting  $g_\infty(r)$  by  $g(r)$  we have

$$g(r) = - \int_{|y|=r} |y|^{2\lambda} v c_{ij} \frac{\partial v}{\partial y_i} \cos(N, y_j) ds - 2\lambda \int_{|y|>r} |y|^{2\lambda-1} v \frac{\partial v}{\partial N} dy$$

$$- 2\lambda \int_{|y|>r} |y|^{2\lambda-1} v (c_{ij} - \delta_{ij}) \frac{\partial v}{\partial y_i} \cos(N, y_j) dy = I_1 + I_2 + I_3 . \quad (4.21)$$

Applying (4.10), (4.13), (4.17) we have

$$g(r) = [1 + o(r^{-\alpha})] f(r) . \quad (4.22)$$

Applying Schwarz inequality to  $I_1$  and applying (4.10), (4.13), (4.14) we have

$$I_1^2 \leq r \int_{|y|=r} |y|^{2\lambda-1} v^2 ds \int_{|y|=r} |y|^{2\lambda} [c_{ij} \frac{\partial v}{\partial y_i} \cos(N, y_j)]^2 ds \leq$$

$$\frac{nf(v)}{n-2+2\lambda} \int_{|y|=r} |y|^{2\lambda} [\frac{\partial v}{\partial N} + |y|^{-\alpha} o(|\text{grad } v|)^2] ds \leq$$

$$[1 + o(r^{-\alpha})] r f(r) / (n-2+2\lambda) \int_{|y|=r} |y|^{2\lambda} (\text{grad } v)^2 ds$$

$$= [1 + o(r^{-\alpha})] [rf(r)f'(r)] / (n-2+2\lambda) . \quad (4.23)$$

Integrate  $I_2$  by parts

$$I_2 = -2\lambda \int_r^\infty \mu^{2\lambda+n-2} d\mu \int_{\omega} v \frac{\partial v}{\partial \mu} d\omega = -\lambda (\mu^{2\lambda+n-2} \int_{\omega} v^2 d\omega)_{\mu=\infty}^\infty$$

$$+ \lambda(2\lambda+n-2) \int_r^\infty \mu^{2\lambda+n-3} d\mu \int_{\omega} v^2 d\omega = (1 - \frac{n}{2} + \beta) \int_{|y|=r} |y|^{2\lambda-1}$$

$$v^2 ds + \beta(1 - \frac{n}{2} + \beta) \int_{|y|=r} |y|^{2\lambda-2} v^2 dy \leq 0 \quad (4.24)$$

the validity of the above formula has used (12) and the relation

$$(\mu^{2\lambda+n-2} \int_{\omega} v^2 d\omega)_{\mu=\infty} = 0 .$$

From (4.15) we have

$$\int_{|y|<r} |y|^{2\lambda-2-2\alpha} v^2 dy \leq \frac{1}{n-2+2\lambda} \int_r^\infty \mu^{-1-2\alpha} d\mu \int_{|y|=r} |y|^{2\lambda} (\text{grad } v)^2 dy$$

$$\leq L_{20} r^{-2\alpha} f(r) . \quad (4.25)$$

From (4.10), (4.13), (4.25) and applying Schwarz inequality to estimate  $I_3$  we have

$$|I_3| \leq L_{21} \int_{|y|>r} |y|^{2\lambda-1-\alpha} |v| |\text{grad } v| dy \leq L_{22} r^{-\alpha} f(r) . \quad (4.26)$$

Substituting (4.22), (4.23), (4.24), (4.26) into (4.21) we have

$$[1 + o(r^{-\alpha})] f(r) \leq \{[1 + o(r^{-\alpha})] (-rf(r)f'(r)) / (n-2+2\lambda)\}^{1/2}$$

or

$$f(r) \leq [1 + O(r^{-\alpha})](-rf'(r))/(n-2+2\lambda) .$$

Integrating we get

$$f(r) \leq f(r_0) \left(\frac{r_0}{r}\right)^{n-2+2\lambda} e^{O(r_0^{-\alpha})} \leq L_{23} r^{2-n-2\lambda} \quad (r > r_0(\Omega, \Gamma))$$

or

$$\int_{|y|>r} |y|^{2-n+2\beta} (\text{grad } v)^2 dy \leq L_{23} r^{-2\beta} . \quad (4.27)$$

Substituting (4.27) into (4.15) we have

$$\int_{|y|=r} |y|^{-n+2\beta} v^2 ds \leq L_{24} r^{-2\beta} . \quad (4.28)$$

Take  $\gamma$  satisfying

$$0 < \gamma < \frac{n}{2} - 1, \quad \gamma < 2\beta .$$

In (4.2) taking  $\lambda = 1 - \frac{n}{2} + \gamma$ ,  $\bar{R} = \infty$ ,  $\theta = 1$ , substituting into (4.1) and applying (4.28) we have

$$\int_{|y|>2R} |y|^{2-n+2\gamma} (\text{grad } v)^2 dy \leq \zeta^2 (\text{grad } v)^2 dy \leq L_{25} \int_{|y|>R} |y|^{-n+2\gamma} v^2 dy \leq L_{26} \int_R^\infty \mu^{-1-4\beta+2\gamma} d\mu \leq L_{27} R^{-4\beta+2\gamma} . \quad (4.29)$$

Note that (4.29) improves (4.13).

Repeating the above process we prove that

$$\int_{|y|>r} |y|^{2-n+2\gamma} (\text{grad } v)^2 dy \leq L_{28} r^{-2\gamma} .$$

Continuing in this way, after a finite number of steps we have

$$\int_{|y|>r} (\text{grad } v)^2 dy \leq L_{29} r^{2-n} . \quad (4.30)$$

We need the following result of [2] extended to the  $n$  dimensional case:

$$\int_{|y|>r} (\text{grad } v)^2 dy \leq L_{30} r^{-2\sqrt{n-1}} . \quad (4.31)$$

Proof. Let  $\bar{S}$  be a closed surface which lies in  $\Omega$  and contains  $\Gamma$  in its interior. Integrating (1) and applying (9) we have

$$\int_{\bar{S}} \rho \frac{\partial \varphi}{\partial x_i} \cos(N, x_i) d\bar{S} = \int_{\Gamma} \rho \frac{\partial \varphi}{\partial N} d\bar{S} = 0 . \quad (4.32)$$

When  $x_h$  changes to  $x_h + \epsilon$ , the changed function will be denoted by adding the subscript  $\epsilon$ . Denote by  $\bar{S}_\epsilon$  the surface  $\bar{S}$  translated by  $(0, \dots, 0, -\epsilon, 0, \dots, 0)$ , then from (4.32) we have

$$\int_{\bar{S}} (\rho \frac{\partial \varphi}{\partial x_i})_{\epsilon} \cos(N, x_i) d\bar{S} = \int_{\bar{S}_{\epsilon}} \rho \frac{\partial \varphi}{\partial x_i} \cos(N, x_i) d\bar{S}_{\epsilon} = 0 .$$

Differentiating with respect to the parameter  $\epsilon$  and let  $\epsilon = 0$  we have

$$\begin{aligned} \int_{\bar{S}} a_{ij} \frac{\partial \varphi}{\partial x_i} \cos(N, x_j) d\bar{S} &= \int_{\bar{S}} a_{ij} \frac{\partial \varphi}{\partial x_i} \cos(N, x_j) d\bar{S} = \\ \int_{\bar{S}} \frac{\partial \varphi}{\partial \epsilon} (\rho \frac{\partial \varphi}{\partial x_i})_{\epsilon} \Big|_{\epsilon=0} \cos(N, x_i) d\bar{S} &= 0 . \end{aligned}$$

After linear transformation ( $\bar{S}$  changes to  $S$ ) we have

$$\int_S c_{ij} \frac{\partial v}{\partial y_i} \cos(N, y_j) dS = 0 . \quad (4.33)$$

From (4.30) we see that we can take  $\lambda = 0$  in (4.21), and it become (by using (4.33))

$$q(r) = - \int_{|y|=r} (v - \bar{v}) c_{ij} \frac{\partial v}{\partial y_i} \cos(N, y_j) dS \quad (4.34)$$

where

$$\bar{v} = \int_{|y|=r} v dS / \text{mes } S .$$

From Wirtinger's inequality

$$\int_{|y|=r} (v - \bar{v})^2 dS \leq \frac{r^2}{n-1} \int_{|y|=r} [(\text{grad } v)^2 - (\frac{\partial v}{\partial N})^2] dS$$

and (4.10), (4.34) we have

$$\begin{aligned} q(r) &\leq \frac{\sqrt{n-1}}{2r} \int_{|y|=r} (v - \bar{v})^2 dS + \frac{r}{2\sqrt{n-1}} \int_{|y|=r} [c_{ij} \frac{\partial v}{\partial y_i} \cos(N, y_j)]^2 dS \leq \\ &\leq \frac{r}{2\sqrt{n-1}} \int_{|y|=r} [(\text{grad } v)^2 - (\frac{\partial v}{\partial N})^2] dS + \\ &\leq \frac{r}{2\sqrt{n-1}} \int_{|y|=r} [(\frac{\partial v}{\partial N})^2 + L_3 r^{-\alpha} (\text{grad } v)^2] dS = \frac{r}{2\sqrt{n-1}} [1 + O(r^{-\alpha})] \int_{|y|=r} (\text{grad } v)^2 dS \\ &= -r[1 + O(r^{-\alpha})]/(2\sqrt{n-1}) q'(r) . \end{aligned}$$

Integrating we have

$$q(r) \leq q(r_0) \frac{r_0}{r} 2\sqrt{n-1} e^{O(r_0^{-\alpha})} \quad (r > r_0(0, \Gamma)) . \quad (4.35)$$

From (4.10), (4.17), (4.35) we obtain (4.31).

Combining (4.30), (4.31) we obtain

$$\int_{|y|>r} (\text{grad } v)^2 dy \leq L_{32} r^{n-4-2\delta} \quad (\delta > 0) .$$

Substituting into (4.15) we have

$$\int_{|y|=r} v^2 ds \leq L_{33} r^{n-3-2\delta} .$$

Hence we have

$$\int_{|y|>R} |y|^{2-n+\delta} v^2 dy = \int_R^\infty r^{2-n+\delta} dr \int_{|y|=r} v^2 ds \leq L_{34} R^{-\delta} .$$

Reverting to the variable  $x$  we have

$$\int_{|x|>R} |x|^{2-n+\delta} v^2 dx \leq L_{35} R^{-\delta} . \quad (4.36)$$

Let  $\psi = \varphi - Ux_1$ , where  $U$  is defined by (11) above, then from (4.36) we have

$$\int_{|x|>R} |x|^{2-n+\delta} (\text{grad } \psi)^2 dx \leq n L_{35} R^{-\delta} .$$

Let  $x$  be the inversion point of  $x$  about the unit sphere and let  $\frac{1}{R} = u$ , the above formula becomes

$$\int_{|x|<u} |x|^{2-n-\delta} (\text{grad}_x \psi)^2 dx \leq n L_{35} u^\delta . \quad (4.37)$$

Fixing a point  $x^0 \in \Omega$ , without loss of generality we can assume temporarily that  $\varphi(x^0) = 0$ , hence  $\psi(x) = -q x_1^0 + \int_{x^0}^x v_h dx_h$ , and combining with (2.17) we have

$$|\psi(x)| \leq L_{36} |x|$$

or

$$|\psi(x)| \leq L_{36} |x|^{-1} . \quad (4.38)$$

From (4.37), (4.38) we have

$$\psi \in W_2^1 \text{ in } |x| < u_0(\Omega, \Gamma)$$

and the following Sobolev decomposition formula is valid [12]:

$$\psi(x) = \int_{|y|<u_0} \zeta(y) \psi(y) dy + \int_{|y|<u_0} |y|^{1-n} w_h(x, y) \frac{\partial \psi}{\partial y} dy \quad (4.39)$$

where  $\zeta$  and  $w_h$  are known bounded functions.

From (4.37) we have

$$\int_{|y|<u} |y|^{1-n} |\text{grad}_y \psi| dy \leq \left[ \int_{|y|<u} |y|^{\delta-n} dy \int_{|y|<u} |y|^{2-n-\delta} (\text{grad}_y \psi)^2 dy \right]^{1/2} .$$

Hence the right hand side of (4.39) is a continuous and bounded function when  $|x| < u_0$ .

Therefore  $|\psi(x)|_{x=\infty}$  exists, and

$$|\psi(x)| \leq L_{39} . \quad (4.40)$$

Denote  $\psi(x) - \psi(\infty)$  still by  $\psi(x)$ . Then from (3.8), (3.22), (4.8), (4.40) we obtain that the solution of (1), (8), (9), (11), (2.16) is bounded in  $E$ .

All the requirements of Leray-Schauder degree theory are satisfied, hence theorem 2 is proved.

## §5. Some properties of subsonic flow.

Theorem 3. The solution of problem (1), (8), (9), (10) is unique in the subsonic range.

Proof. From (2) we obtain that, the difference  $\bar{\varphi} = \varphi^{(2)} - \varphi^{(1)}$  of two solutions  $\varphi^{(1)}, \varphi^{(2)}$  satisfies the following equation

$$a_{ij}(u_1^{(2)}, \dots, u_n^{(2)}) \frac{\partial^2 \bar{\varphi}}{\partial x_i \partial x_j} + b_k \frac{\partial \bar{\varphi}}{\partial x_k} = 0 \quad (5.1)$$

where  $u_i^{(1)} = \frac{\partial \varphi^{(1)}}{\partial x_i}$ ,  $u_i^{(2)} = \frac{\partial \varphi^{(2)}}{\partial x_i}$  and

$$b_k = [a_{ij}(u_1^{(2)}, \dots, u_k^{(2)}, u_{k+1}^{(1)}, \dots, u_n^{(1)}) - a_{ij}(u_1^{(2)}, \dots, u_{k-1}^{(2)}, u_k^{(1)}, \dots, u_n^{(1)})] / [u_k^{(2)} - u_k^{(1)}] \frac{\partial^2 \varphi^{(1)}}{\partial x_i \partial x_j} \quad (5.2)$$

From (8) we have

$$\frac{\partial \bar{\varphi}}{\partial N} \Big|_{\Gamma} = 0 \quad . \quad (5.3)$$

From the last section we have:  $\varphi^{(1)}, \varphi^{(2)}$  (after subtraction of a suitable constant) satisfy (2.6), hence we have

$$\bar{\varphi}(\infty) = 0 \quad . \quad (5.4)$$

From (5.1), (5.2) by applying the maximum principle [5] we have, when  $\bar{\varphi}$  is not a constant, then it cannot take a positive maximum value or a negative minimum value in  $\Omega + \Gamma$ , so combining with (5.4) we have  $\bar{\varphi} = 0$ . This proves the theorem.

Theorem 4. There exists a positive constant  $q_c^\infty$ , such that when  $0 < U < q_c^\infty$ , the

solution of problem (1), (8), (9), (10) exists in the subsonic range (this solution is unique by theorem 3). Moreover, the function  $Q$  defined by (12) satisfy

$Q(U) \in C[0, q_c^\infty)$ , and

$$\lim_{U \rightarrow q_c^\infty} Q(U) = q_c^\infty . \quad (5.5)$$

The proof is divided into several points as follows:

1) Let  $\{\varphi_n\}$  be any sequence of solutions of (1), (8), (9), (11), (12), (13) with  $Q, U$  substituted by  $Q_n, U_n$ . If  $\sup Q_n < q_c$ , then from §2, iii we have that  $\|\varphi_n\|$  is uniformly bounded in  $E$ ,  $\{\varphi_n\}$  is compact by theorem 1, hence a limit function  $\varphi$  exists, it is easy to prove that  $\varphi$  is a solution of (1), (8), (9), (11), (12), (13) also and its corresponding  $Q$  is a limit point of the sequence  $\{Q_n\}$ .

2) Take any solution  $\varphi$  of (1), (8), (9), (11), (12), (13), we define  $U = U(Q, \varphi)$  by (11). Fixed  $Q (0 < Q < q_c)$  and denote the infimum of all  $U(Q, \varphi)$  by  $V(Q)$ , i.e. a solution sequence  $\{\varphi_n\}$  exists such that  $U(Q, \varphi_n) \rightarrow V(Q)$ . By 1), there exists a limit function  $\varphi = \varphi(Q)$ , which is the solution of (1), (8), (9), (11), (12), (13) (where we substitute for  $U$  by  $V$  in (11)).

3) Let  $0 < Q_0 < q_c$ , we prove that the interval  $[0, V(Q_0)]$  is covered by the set of all  $(U, \varphi) (0 < Q < Q_0)$ .

If this is false, i.e. constant  $U_0$  exists such that  $0 < U_0 < V(Q_0)$  and the problem (1), (8), (9), (10) (where in (10) we substitute for  $U$  by  $U_0$ ), has no solution in  $0 < Q < Q_0$ .

If there is a positive number sequence  $\varepsilon_n$  with  $\varepsilon_n \rightarrow 0 (n \rightarrow \infty)$  such that (1), (8), (9), (10), (where in (10) we substitute for  $U$  by  $U_0 - \varepsilon_n$ ), has a sequence of solution  $\varphi_n$  in  $0 < Q < Q_0$ , then by 1), the limit function  $\varphi$  is the solution of (1), (8), (9), (10), where in (10) we substitute for  $U$  by  $U_0$ , in  $0 < Q < Q_0$ , and this contradicts the above hypothesis.

Hence a constant  $U_1$  exists, satisfying  $0 < U_1 < U_0$ , such that the problem (1), (8), (9), (10) has no solution in  $0 < Q < Q_0$  when  $U_1 < U < U_0$ .

Now apply Leray-Schauder degree theory\* to the set  $-\infty < U < \frac{1}{2}(U_0 + U_1)$  in the space  $E$  (for the relation of the element  $\varphi \in E$  and  $U$  see (2.6)). We obtain a solution to problem (1), (8), (9), (11), (12) for any given  $0 < Q < Q_0$  and this solution possesses the following property:  $U$  defined from (9), (10) satisfies  $0 < U < U_0$ . Hence

$$\inf U(Q_0) < V(Q_0) .$$

This contradicts the definition of  $V(Q_0)$ , which proves 3.

4) The function  $V(Q)$  is strictly monotone increasing in the interval  $[0, q_c]$ .

Otherwise there are  $0 < Q_1 < Q_2 < q_c$  such that  $V(Q_1) > V(Q_2)$ . From 3) we have, when  $0 < Q < Q_1$  the set of  $U(Q, \varphi)$  covers the interval  $[0, V(Q_1)]$ , hence we have  $Q_0 \in [0, Q_1]$  and a  $\varphi_0$  such that  $U(Q_0, \varphi_0) = V(Q_2)$ , contrary to theorem 3.

$V(Q)$  is strictly monotone increasing and has supremum  $q_c$ , hence  $V(q_c - 0)$  exists.

5) Let

$$q_c^\infty = V(q_c - 0) .$$

From 3) we have, the set of  $U(Q)$  ( $0 < Q < q_c$ ) covers the interval  $0 < U < q_c^\infty$ . In other words, the solution of problem (1), (8), (9), (10) in the subsonic range exists when  $0 < U < q_c^\infty$ .

The above solution is also unique by theorem 3, hence the function

$$Q(U), \quad 0 < U < q_c^\infty ,$$

is determined uniquely.

For fixed  $U_0$  in  $[0, q_c^\infty]$ , it lies in the number set  $U(Q(U_0), \varphi)$ , hence we have

$$V(Q(U_0)) < U_0 . \quad (5.6)$$

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\*From §2, iii, but substitute  $Q$  by  $Q_0$ , positive constant  $K$  exists such that the solution of (2.13) satisfying  $\|\varphi\| \leq K$ . Take  $F$  be the part of sphere  $\|\varphi\| \leq K+1$  satisfying

$$-\infty < U < \frac{1}{2}(U_0 + U_1) \quad (\varphi = UX_1 + \varphi_1, D\varphi_1(\infty) = 0)$$

then (2.13) has no solution in the neighborhood of the boundary of  $F$ . From §2, ii we have, the degree of solution is 1 in  $F$  when  $\alpha = 0$ , combine with §2, i and applying Leray-Schauder degree theory we obtain that, when  $0 \leq \alpha \leq 1$ , (1), (8), (9), (11), (12) in  $F$  has at least one solution.

6) From the definition of  $q_c^\infty$  there exists  $Q_n + q_c = 0$ , such that  $V(Q_n) + q_c^\infty$ . If  $U$  satisfies  $0 < U < V(Q_n)$  and  $Q(U) > Q_n$ . Combining with (5.6) we have  $U > V(Q_n)$ , this contradicts  $U < V(Q_n)$ . Hence when  $0 < U < V(Q_n)$ . we have

$$0 < Q(U) < Q_n + q_c^\infty. \quad (5.7)$$

7) Let  $\{U_m\}$  satisfy  $0 < U_m < V(Q_n)$ ,  $U_m \rightarrow U_0$ ,  $Q(U_m) \rightarrow Q_0$ . From (5.7) by applying 1) we obtain that the limit function  $\varphi$  is the solution of (1), (8), (9), (10), where  $U$ ,  $Q$  are given by  $U_0$ ,  $Q_0$  in (10), (12) respectively. Applying theorem 3 we have  $Q_0 = Q(U_0)$ , in other words  $Q(U) \in C[0, V(Q_n)]$ . Letting  $n \rightarrow \infty$  we have

$$Q(U) \in C[0, q_c^\infty].$$

8) From the above points we see easily that  $\lim_{\substack{U \rightarrow q_c^\infty \\ U + q_c = 0}} Q(U) = q_c^\infty$ . then the functional value of the continuous function  $Q(U)$  oscillates finitely in the interval  $(Q_0, q_c^\infty)$ , and every point in  $[Q_0, q_c^\infty]$  is the limit point of  $Q(U)$  when  $U + q_c^\infty = 0$ . Hence we have  $U_m + q_c^\infty, U'_m + q_c^\infty$  such that  $Q(U_m) \rightarrow Q_0$ ,  $Q(U'_m) \rightarrow \frac{1}{2}(Q_0 + q_c^\infty)$ . By 1), the limit functions  $\varphi, \varphi'$  of corresponding solution sequences  $\{\varphi_m\}, \{\varphi'_m\}$  are solutions of (1), (8), (9), (11), (12), (13), where  $Q$  is given by  $Q_0$  and  $\frac{1}{2}(Q_0 + q_c^\infty)$  respectively. Hence  $U(Q_0) = q_c^\infty = U\left(\frac{Q_0 + q_c^\infty}{2}\right)$  or the problem (1), (8), (9), (10) (substitute  $U$  by  $q_c^\infty$ ) has at least two solutions in the subsonic range, which contradicts theorem 3, hence (5.5) is true.

The theorem is proved completely.

Condition (11) restricts the incoming flow is in the positive  $x_1$  direction. Now remove this restriction by only assuming

$$u^\infty = (u_1^\infty, \dots, u_n^\infty) \quad (5.8)$$

is given. Because (1), (8) is invariant under axis rotation, hence by theorems 3, 4 we have, the solution of (1), (8), (9), (5.8) exists and is unique. Let

$$\varphi = u_1^\infty x_1 + \psi \quad (5.9)$$

and regard  $\psi$  as a solution of the linear problem, by theorem 1 we have

$$\|\psi\|_{n-2, 2+\tau} \leq L_{39} \|u^\infty\|. \quad (5.10)$$

Now prove that the solution of (1), (8), (9), (5.8) depends continuously on  $u$  by the following sense.

Theorem 5. There exists a positive constant  $L_{40}$ , such that for any two vectors  $u^{\infty,1}$ ,  $u^{\infty,2}$  in the class

$$|u^{\infty}| \leq v\left(\frac{\rho}{|u^{\infty}|} u_1^{\infty}, \dots, \frac{\rho}{|u^{\infty}|} u_n^{\infty}\right)$$

(the definition of function  $v$  see theorem 4) the corresponding solutions  $\varphi^{(1)}, \varphi^{(2)}$  of (1), (8), (9), (5.8) satisfy

$$|\psi^{(2)} - \psi^{(1)}|_{n-2,2+\tau} \leq L_{40} |u^{\infty,2} - u^{\infty,1}| \quad (5.11)$$

where the relation of  $\varphi$  and  $\psi$  is given by (5.9).

Proof. Let  $\bar{\varphi} = \psi^{(2)} - \psi^{(1)}$ , from (2) we obtain that  $\bar{\varphi}$  satisfies (5.1), (5.2).

From (8) we have

$$\frac{\partial^2}{\partial N} \bar{\varphi} \Big|_{\Gamma} = -(u_1^{\infty,2} - u_1^{\infty,1}) \cos(N, x_1) \Big|_{\Gamma}.$$

Applying (6), (5.10) it is easy to show that the  $b_k$  defined from (5.2) satisfy conditions

(1.7), (1.55), (1.59), and from (5.10) we get (5.4). Hence from theorem 1 we have (5.11).

This completes the proof.

Theorem 6. For every non zero subsonic flow around a given profile,  $q_{\max}$  cannot be taken on  $\Omega$  or at  $\infty$ . In other words,  $q_{\max}$  can only be taken on  $\Gamma$ . And

$$q_{\max} \leq L_{41} |u^{\infty}|. \quad (5.12)$$

Proof. If  $q_{\max} > 0$  is taken by  $P \in \Omega$ , then after a rotation of axis we arrive at  $\text{grad } u|_P = (q_{\max}, 0, \dots, 0)$  and (1), (8) remain unchanged, hence  $u_1$  takes the maximum value at  $P$ . And  $u_1$  satisfies the elliptic equation (3.1), hence from the maximum principle [5] we have,  $u_1$  is a constant, i.e.  $u_1 = q_{\max}$ , hence  $u_2 = \dots = u_n = 0$ . From (8) we have  $q_{\max} = 0$ . This is a contradiction.

From (5.10) and applying the result of [2], we have that  $q_{\max}$  cannot be taken at  $\infty$ .

(5.12) is a special case of (5.11) with  $u^{\infty,2} = 0$ . The proof of the theorem is thus complete.

Theorem 7.  $u|_{x=\infty}$  exists for any subsonic flow in the neighborhood of  $\infty$  and (4.8) is valid. An the subsonic flow in the whole space is uniform.

Proof. For subsonic flow in the neighborhood of  $\infty$ , because (4.4), (4.5) and (4.6) are valid, hence the existence of  $u|_{x=\infty}$  and the validity of (4.8) follow from [3].

Concerning subsonic flow in the whole space,  $u|_{x=\infty}$  exists by the above argument, hence it is a special case of flow around profile, i.e. (4.32) is true for every closed surface. Hence (5.10) is valid, and we can apply theorem 6, i.e. except  $q$  is a constant,  $q_\infty$  cannot be taken by any finite point and  $\infty$ . And when  $q$  is a constant,  $u$  is a constant vector by theorem 6, i.e. the flow is uniform.

The theorem is thus proved.

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